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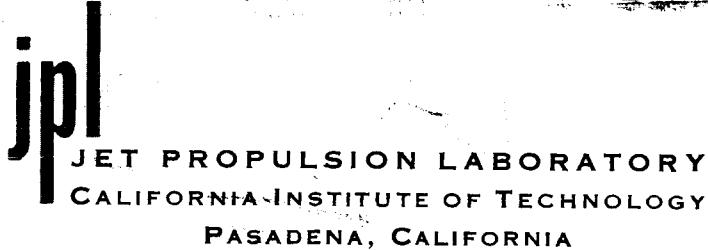
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Analysis of Short, Tapered Conical Shells

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Analysis of Short, Tapered Conical Shells

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ABSTRACT

This Report presents an approximate solution of a tapered conical shell segment under axisymmetric loading. The solution is based on the assumption that the ratio Δ/y_0 is small compared to one. Part I presents the case of edge moment and shear loads. Part II presents the case of uniform pressure and vertical edge forces.

I. AXISYMMETRIC EDGE LOADING

The following is the solution for a short conical shell with a linear thickness variation under axisymmetric edge loading. The solution applies to shells for which the ratio Δ/y_0 is small compared to one. Shells with either a positive or a negative thickness variation are considered. Figure 1 shows the notation and sign convention.

The equations that describe a shell of revolution can be reduced to two second-order differential equations. These equations are as follows (Ref. 1, Eqs. 315, 316):

$$\begin{aligned} \frac{r_2}{r_1^2} \frac{d^2 U}{d\phi^2} + \frac{1}{r_1} \left[\frac{d}{d\phi} \left(\frac{r_2}{r_1} \right) + \frac{r_2}{r_1} \cot \phi - \frac{r_2}{r_1 h} \frac{dh}{d\phi} \right] \frac{dU}{d\phi} \\ - \frac{1}{r_1} \left[\frac{r_1}{r_2} \cot^2 \phi - \nu - \frac{\nu}{h} \frac{dh}{d\phi} \cot \phi \right] U = E h V \end{aligned} \quad (1)$$

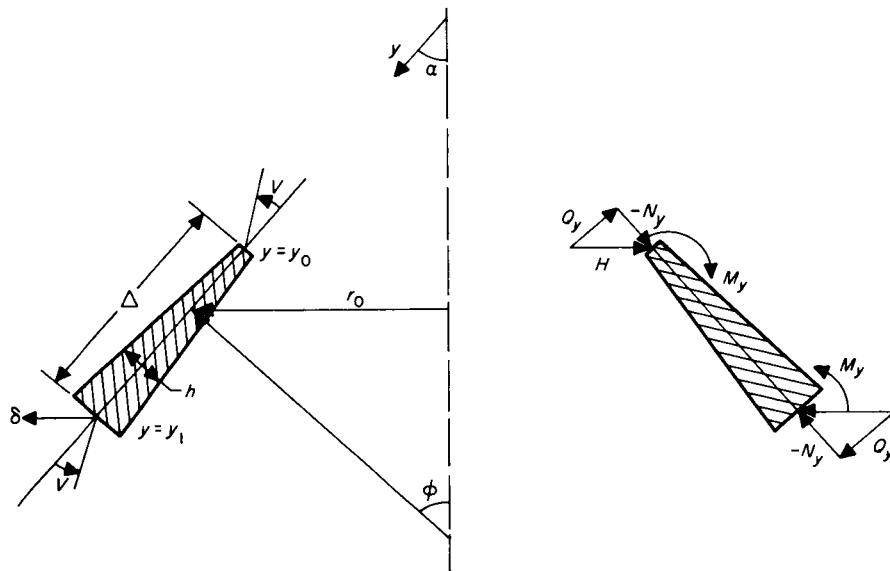


Fig. 1. Notation and sign convention

$$\begin{aligned}
 & \frac{r_2}{r_1^2} \frac{d^2V}{d\phi^2} + \frac{1}{r_1} \left[\frac{d}{d\phi} \left(\frac{r_2}{r_1} \right) + \frac{r_2}{r_1} \cot \phi + \frac{3r_2}{r_1 h} \frac{dh}{d\phi} \right] \frac{dV}{d\phi} \\
 & - \frac{1}{r_1} \left[\frac{r_1}{r_2} \cot^2 \phi + \nu - \frac{3\nu \cot \phi}{h} \frac{dh}{d\phi} \right] V = \frac{-U}{D} \quad (2)
 \end{aligned}$$

These equations can be applied to a conical shell with the following substitutions:

$$\begin{aligned}
 r_1 d\phi &= dy & \frac{d}{d\phi} &= r_1 \frac{d}{dy} \\
 \frac{d^2}{d\phi^2} &= r_1^2 \frac{d^2}{dy^2} & r_1 &\rightarrow \infty
 \end{aligned}$$

Equations (1) and (2) then become

$$y \frac{d^2 U}{dy^2} + \frac{dU}{dy} \left(1 - y \frac{h'}{h} \right) + U \left(\nu \frac{h'}{h} - \frac{1}{y} \right) = \frac{E h V}{\tan \alpha} \quad (3)$$

$$y \frac{d^2 V}{dy^2} + \frac{dV}{dy} \left(1 + 3y \frac{h'}{h} \right) + V \left(3\nu \frac{h'}{h} - \frac{1}{y} \right) = \frac{-U}{D \tan \alpha} \quad (4)$$

where

$$h' = \frac{dh}{dy}$$

Let

$$U = \bar{U} \sqrt{\frac{h}{y}}$$

$$V = \frac{\bar{V}}{\sqrt{h^3 y}}$$

$$\frac{dh}{dy} = \text{constant (linear thickness variation)}$$

Then Eqs. (3) and (4) become

$$\frac{d^2 \bar{U}}{dy^2} + \bar{U} \left[-\frac{3}{4} \left(\frac{h'}{h} \right)^2 - \frac{3}{4y^2} + \frac{h'}{hy} \left(\frac{1}{2} + \nu \right) \right] = \frac{E \bar{V}}{hy \tan \alpha} \quad (5)$$

$$\frac{d^2 \bar{V}}{dy^2} + \bar{V} \left[-\frac{3}{4} \left(\frac{h'}{h} \right)^2 - \frac{3}{4y^2} - \frac{3h'}{hy} \left(\frac{1}{2} - \nu \right) \right] = -\frac{\bar{U} 12 (1 - \nu^2)}{E hy \tan \alpha} \quad (6)$$

Equations (5) and (6) can be reduced to one second-order equation by multiplying Eq. (6) by a complex constant, $\lambda = a + ib$, and adding the two equations.

$$\begin{aligned} \frac{d^2}{dy^2} (\bar{U} + \lambda \bar{V}) &= \bar{U} \left[\frac{3}{4} \left(\frac{h'}{h} \right)^2 + \frac{3}{4y^2} - \frac{h'}{hy} \left(\frac{1}{2} + \nu \right) \right] + \frac{E \bar{V}}{hy \tan \alpha} \\ &\quad + \lambda \bar{V} \left[\frac{3}{4} \left(\frac{h'}{h} \right)^2 + \frac{3}{4y^2} + \frac{3h'}{hy} \left(\frac{1}{2} - \nu \right) \right] - \frac{\lambda \bar{U} 12 (1 - \nu^2)}{E hy \tan \alpha} \end{aligned}$$

$$\frac{d^2}{dy^2} (\bar{U} + \lambda \bar{V}) = \left[\frac{3}{4} \left(\frac{h'}{h} \right)^2 + \frac{3}{4y^2} \right] (\bar{U} + \lambda \bar{V})$$

$$+ \left\{ \frac{\bar{U} + \bar{V}}{h'} \left[\frac{3h' \left(\frac{1}{2} - \nu \right) \lambda + \frac{E}{\tan \alpha}}{-h' \left(\frac{1}{2} + \nu \right) - \frac{12\lambda(1-\nu^2)}{E \tan \alpha}} \right] \right\} \left[-\frac{h'}{hy} \left(\frac{1}{2} + \nu \right) - \frac{12\lambda(1-\nu^2)}{Ehy \tan \alpha} \right]$$

Let

$$\lambda = \frac{3h' \left(\frac{1}{2} - \nu \right) \lambda + \frac{E}{\tan \alpha}}{-h' \left(\frac{1}{2} + \nu \right) - \frac{12\lambda(1-\nu^2)}{E \tan \alpha}}$$

Solving for λ , we get

$$\lambda = \frac{h'E \tan \alpha}{12(1+\nu)} \left[-1 + i \sqrt{\frac{12(1+\nu)}{(1-\nu)(h')^2 \tan^2 \alpha} - 1} \right]$$

Introducing a new variable $W = \bar{U} + \lambda \bar{V}$, Eqs. (5) and (6) become

$$\frac{d^2W}{dy^2} + W \left[\frac{h'}{hy} \left(-\frac{1}{2} + 2\nu \right) - \frac{3}{4} \left(\frac{h'}{h} \right)^2 - \frac{3}{4y^2} + \frac{h'}{hy} (1-\nu)i \sqrt{\frac{12(1+\nu)}{(1-\nu)(h')^2 \tan^2 \alpha} - 1} \right] = 0 \quad (7)$$

The relative order of magnitude of the terms inside the bracket of Eq. (7) can be examined by non-dimensionalizing the coordinate system. Let

$$\bar{h} = \frac{h}{h_0}, \quad \Delta = y_1 - y_0, \quad \bar{s} = \frac{y - y_0}{\Delta}$$

$$n = \rho \frac{\Delta}{h_0} = \frac{d\bar{h}}{ds} = \bar{h}'$$

Then Eq. (7) becomes

$$\frac{d^2\bar{W}}{ds^2} + \left[\frac{\bar{h}' \left(-\frac{1}{2} + 2\nu \right)}{\bar{h} \left(1 + \frac{\Delta s}{y_0} \right)} \frac{\Delta}{y_0} - \frac{3}{4} \left(\frac{\bar{h}'}{\bar{h}} \right)^2 - \frac{3}{4} \frac{1}{\left(1 + \frac{\Delta s}{y_0} \right)^2} \left(\frac{\Delta}{y_0} \right)^2 \right. \\ \left. + \frac{\bar{h}'(1-\nu)i}{\bar{h} \left(1 + \frac{\Delta s}{y_0} \right)} \sqrt{\frac{12(1+\nu)\Delta^4}{(1-\nu)(\bar{h}')^2 \tan^2 \alpha y_0^2 h_0^2} - \left(\frac{\Delta}{y_0} \right)^2} \right] \bar{W} = 0 \quad (8)$$

The relative order of magnitude of the first three terms inside the bracket is

$$\frac{\bar{h}'\Delta}{\bar{h}y_0 \left(1 + \frac{\Delta s}{y_0} \right)} \left(-\frac{1}{2} + 2\nu \right) = \left(-\frac{1}{2} + 2\nu \right) \left[0 \left(\frac{\Delta}{y_0} \right) \right] < 0 \left(\frac{\Delta}{y_0} \right) \\ \frac{3}{4} \left(\frac{\bar{h}'}{\bar{h}} \right)^2 = O(1) \\ \frac{3}{4} \left(\frac{\Delta}{y_0} \right)^2 = O \left(\frac{\Delta}{y_0} \right)^2$$

If Δ/y_0 is assumed to be small compared to one, Eq. (8) becomes

$$\frac{d^2\bar{W}}{ds^2} + \left[-\frac{3}{4} \left(\frac{\bar{h}'}{\bar{h}} \right)^2 + \frac{\bar{h}'}{\bar{h}} (1-\nu)i \sqrt{\frac{12(1+\nu)\Delta^4}{(\bar{h}')^2(1-\nu)\tan^2 \alpha y_0^2 h_0^2} - \left(\frac{\Delta}{y_0} \right)^2} \right] \bar{W} = 0 \quad (9)$$

The requirement that Δ/y_0 be small compared to one is the basic restriction on this analysis. Further simplification of Eq. (9) can be obtained by making the following substitutions:

$$x = \bar{h} = 1 + ns$$

$$K = \frac{1-\nu}{n} \sqrt{\frac{12(1+\nu)\Delta^4}{(1-\nu)n^2 \tan^2 \alpha y_0^2 h_0^2} - \left(\frac{\Delta}{y_0} \right)^2}$$

Equation (9) then becomes

$$x^2 \frac{d^2 W}{dx^2} + \left[-\frac{3}{4} + iKx \right] W = 0 \quad (10)$$

The parameter K can be positive or negative, depending on the relative size of h_1 and h_0 . The sign of K will affect the solution; therefore, each case will be treated separately. In Case I, $h_1 < h_0$; therefore, $n =$ negative. In Case II, $h_1 > h_0$; therefore, $n =$ positive.

Case I

Let $k = |K|$. Equation (10) then becomes

$$x^2 \frac{d^2 W}{dx^2} + \left[-\frac{3}{4} - ikx \right] W = 0 \quad (11)$$

This equation is of the form

$$x^2 y'' + x(a + 2bx')y' + [C + dx^{2s} - b(1-a-r)x^r + b^2 x^{2r}]y = 0$$

whose solution is given by (Ref. 2, pp. 166-169):

$$y = x^{\left(\frac{1-a}{2}\right)} e^{-\left(\frac{bx^r}{r}\right)} \sum_p \left(\frac{\sqrt{d}}{s} x^s \right)$$

where

$$p = \frac{1}{s} \sqrt{\left(\frac{1-a}{2}\right)^2 - C}$$

It follows that the solution to Eq. (10) is given by

$$\begin{aligned} W = & \frac{\bar{x}}{2\sqrt{2k}} [C_1 \text{ber}_2 \bar{x} + C_2 \text{bei}_2 \bar{x} + C_3 \text{ker}_2 \bar{x} + C_4 \text{kei}_2 \bar{x} \\ & + i(C_1 \text{bei}_2 \bar{x} - C_2 \text{ber}_2 \bar{x} + C_3 \text{kei}_2 \bar{x} - C_4 \text{ker}_2 \bar{x})] \end{aligned} \quad (12)$$

where

$$\bar{x} = 2\sqrt{k}x^{\frac{1}{2}}$$

Since $\bar{W} = \bar{U} + \lambda \bar{V} = \bar{U} + (a + ib) \bar{V}$, the real and imaginary parts can be equated to obtain:

$$\bar{V} = \frac{\bar{x}}{2b\sqrt{2k}} [C_1 \operatorname{bei}_2 \bar{x} - C_2 \operatorname{ber}_2 \bar{x} + C_3 \operatorname{kei}_2 \bar{x} - C_4 \operatorname{ker}_2 \bar{x}] \quad (13)$$

$$\begin{aligned} \bar{U} = & \frac{\bar{x}}{2\sqrt{2k}} \left[C_1 \left(\operatorname{ber}_2 \bar{x} - \frac{a}{b} \operatorname{bei}_2 \bar{x} \right) + C_2 \left(\operatorname{bei}_2 \bar{x} + \frac{a}{b} \operatorname{ber}_2 \bar{x} \right) \right. \\ & \left. + C_3 \left(\operatorname{ker}_2 \bar{x} - \frac{a}{b} \operatorname{kei}_2 \bar{x} \right) + C_4 \left(\operatorname{kei}_2 \bar{x} + \frac{a}{b} \operatorname{ker}_2 \bar{x} \right) \right] \end{aligned} \quad (14)$$

Expressions for M_y and δ can be written as follows (Ref. 1, Sections 127, 128):

$$M_y = - \frac{D}{\Delta} \left(\frac{dV}{ds} + \nu \frac{\Delta}{y} V \right)$$

$$\delta = - \frac{y \sin \alpha}{Eh \Delta} \left(\frac{dU}{ds} - \nu \frac{\Delta}{y} U \right)$$

Remembering that

$$U = \bar{U} \sqrt{\frac{h}{y}}, \quad V = \frac{\bar{V}}{\sqrt{h^3 y}}, \quad y = y_0 \left[1 + \frac{\bar{x}^2 - 4k}{4\beta} \right]$$

and letting

$$\eta = \sqrt{h_0 y_0 \left(1 + \frac{\bar{x}^2 - 4k}{4\beta} \right)}$$

$$\xi = \frac{2}{\bar{x}} + \frac{\bar{x}(1 - 2\nu)}{4\beta \left(1 + \frac{\bar{x}^2 - 4k}{4\beta} \right)}$$

$$\psi = \frac{2}{\bar{x}} - \frac{\bar{x}(1+2\nu)}{4\beta \left(1 + \frac{\bar{x}^2 - 4k}{4\beta}\right)}$$

the quantities of interest become

$$V = \frac{4k}{\sqrt{2}bh_0\bar{x}^2\eta} [C_1 \operatorname{bei}_2 \bar{x} - C_2 \operatorname{ber}_2 \bar{x} + C_3 \operatorname{kei}_2 \bar{x} - C_4 \operatorname{ker}_2 \bar{x}]$$

$$U = \frac{h_0\bar{x}^2}{4\sqrt{2}k\eta} \left[C_1 \left(\operatorname{ber}_2 \bar{x} - \frac{a}{b} \operatorname{bei}_2 \bar{x} \right) + C_2 \left(\operatorname{bei}_2 \bar{x} + \frac{a}{b} \operatorname{ber}_2 \bar{x} \right) \right. \\ \left. + C_3 \left(\operatorname{ker}_2 \bar{x} - \frac{a}{b} \operatorname{kei}_2 \bar{x} \right) + C_4 \left(\operatorname{kei}_2 \bar{x} + \frac{a}{b} \operatorname{ker}_2 \bar{x} \right) \right]$$

$$M_y = \frac{Eh_0\rho\bar{x}^3}{96\sqrt{2}(1-\nu^2)kb\eta} [C_1(\xi \operatorname{bei}_2 \bar{x} - \operatorname{bei}'_2 \bar{x}) + C_2(-\xi \operatorname{ber}_2 \bar{x} + \operatorname{ber}'_2 \bar{x}) \\ + C_3(\xi \operatorname{kei}_2 \bar{x} - \operatorname{kei}'_2 \bar{x}) + C_4(-\xi \operatorname{ker}_2 \bar{x} + \operatorname{ker}'_2 \bar{x})]$$

$$\delta = - \frac{\sqrt{2}\rho k \eta \sin \alpha}{Eh_0^2\bar{x}} \left\{ C_1 \left[\psi \left(\operatorname{ber}_2 \bar{x} - \frac{a}{b} \operatorname{bei}_2 \bar{x} \right) + \operatorname{ber}'_2 \bar{x} - \frac{a}{b} \operatorname{bei}'_2 \bar{x} \right] \right. \\ \left. + C_2 \left[\psi \left(\operatorname{bei}_2 \bar{x} + \frac{a}{b} \operatorname{ber}_2 \bar{x} \right) + \operatorname{bei}'_2 \bar{x} + \frac{a}{b} \operatorname{ber}'_2 \bar{x} \right] \right. \\ \left. + C_3 \left[\psi \left(\operatorname{ker}_2 \bar{x} - \frac{a}{b} \operatorname{kei}_2 \bar{x} \right) + \operatorname{ker}'_2 \bar{x} - \frac{a}{b} \operatorname{kei}'_2 \bar{x} \right] \right. \\ \left. + C_4 \left[\psi \left(\operatorname{kei}_2 \bar{x} + \frac{a}{b} \operatorname{ker}_2 \bar{x} \right) + \operatorname{kei}'_2 \bar{x} + \frac{a}{b} \operatorname{ker}'_2 \bar{x} \right] \right\}$$

Case II

In this case, $h_1 > h_0$; therefore, n = positive. Equation (10) is

$$x^2 \frac{d^2 W}{dx^2} + \left[-\frac{3}{4} + iKx \right] W = 0$$

Making the same substitutions as before, namely $\bar{x} = 2\sqrt{K}x^{1/2}$, the solution becomes

$$W = \frac{\sqrt{i}\bar{x}}{2\sqrt{K}} [AJ_2(i^{1/2}\bar{x}) + BY_2(i^{1/2}\bar{x})] \quad (15)$$

$$i^{1/2} = (-i)^{3/2}$$

and

$$J_2(i^{3/2}\bar{x}) = \text{ber}_2\bar{x} + i \text{bei}_2\bar{x}$$

It can be shown that

$$J_2(i^{1/2}\bar{x}) = \text{ber}_2\bar{x} - i \text{bei}_2\bar{x}$$

The function $Y_2(i^{1/2}\bar{x})$ is defined as follows (Ref. 2, p. 157):

$$\begin{aligned} Y_2(i^{1/2}\bar{x}) &= \frac{2}{\pi} \left[\left(\log \frac{i^{1/2}\bar{x}}{2} + \gamma \right) J_2(i^{1/2}\bar{x}) - \frac{1}{2} \sum_{K=0}^1 (1-K)! \frac{(i^{1/2}\bar{x})^{2K-2}}{K!} \right. \\ &\quad \left. + \frac{1}{2} \sum_{K=0}^{\infty} (-1)^{K+1} \frac{\left[\phi_K + \phi_{K+2} \right] \left(\frac{i^{1/2}\bar{x}}{2} \right)^{2K+2}}{K! (2+K)!} \right] \end{aligned}$$

It can be shown that the following relation exists:

$$Y_2(i^{1/2}\bar{x}) = -\frac{2}{\pi} \text{ker}_2\bar{x} + \text{bei}_2\bar{x} + i \left(\frac{2}{\pi} \text{kei}_2\bar{x} + \text{ber}_2\bar{x} \right)$$

Constants A and B are complex, so we can write W in its real and imaginary parts with four arbitrary real constants:

$$W = \frac{\bar{x}}{2\sqrt{2K}} \left\{ A_1 [\operatorname{ber}_2 \bar{x} + \operatorname{bei}_2 \bar{x} + i(\operatorname{ber}_2 \bar{x} - \operatorname{bei}_2 \bar{x})] \right.$$

$$+ A_2 [-\operatorname{ber}_2 \bar{x} + \operatorname{bei}_2 \bar{x} + i(\operatorname{ber}_2 \bar{x} + \operatorname{bei}_2 \bar{x})]$$

$$+ A_3 \left[-\operatorname{ber}_2 \bar{x} + \operatorname{bei}_2 \bar{x} - \frac{2}{\pi} \operatorname{ker}_2 \bar{x} - \frac{2}{\pi} \operatorname{kei}_2 \bar{x} \right.$$

$$\left. + i \left(-\frac{2}{\pi} \operatorname{ker}_2 \bar{x} + \operatorname{bei}_2 \bar{x} + \frac{2}{\pi} \operatorname{kei}_2 \bar{x} + \operatorname{ber}_2 \bar{x} \right) \right]$$

$$+ A_4 \left[\frac{2}{\pi} \operatorname{ker}_2 \bar{x} - \operatorname{bei}_2 \bar{x} - \frac{2}{\pi} \operatorname{kei}_2 \bar{x} - \operatorname{ber}_2 \bar{x} \right.$$

$$\left. + i \left(-\frac{2}{\pi} \operatorname{ker}_2 \bar{x} + \operatorname{bei}_2 \bar{x} - \frac{2}{\pi} \operatorname{kei}_2 \bar{x} - \operatorname{ber}_2 \bar{x} \right) \right] \}$$

Since

$$\bar{U} + (a + ib)\bar{V} = W$$

then,

$$\bar{V} = \frac{\bar{x}}{2b\sqrt{2K}} \left[A_1 (\operatorname{ber}_2 \bar{x} - \operatorname{bei}_2 \bar{x}) + A_2 (\operatorname{ber}_2 \bar{x} + \operatorname{bei}_2 \bar{x}) \right.$$

$$+ A_3 \left(-\frac{2}{\pi} \operatorname{ker}_2 \bar{x} + \operatorname{bei}_2 \bar{x} + \frac{2}{\pi} \operatorname{kei}_2 \bar{x} + \operatorname{ber}_2 \bar{x} \right)$$

$$\left. + A_4 \left(-\frac{2}{\pi} \operatorname{ker}_2 \bar{x} + \operatorname{bei}_2 \bar{x} - \frac{2}{\pi} \operatorname{kei}_2 \bar{x} - \operatorname{ber}_2 \bar{x} \right) \right]$$

Since

$$\bar{U} = Re W - a\bar{V}$$

$$\begin{aligned}\bar{U} = \frac{\bar{x}}{2\sqrt{2K}} & \left\{ A_1 \left[\left(1 - \frac{a}{b}\right) \text{ber}_2 \bar{x} + \left(1 + \frac{a}{b}\right) \text{bei}_2 \bar{x} \right] \right. \\ & + A_2 \left[- \left(1 + \frac{a}{b}\right) \text{ber}_2 \bar{x} + \left(1 - \frac{a}{b}\right) \text{bei}_2 \bar{x} \right] \\ & + A_3 \left[- \left(1 + \frac{a}{b}\right) \left(\text{ber}_2 \bar{x} + \frac{2}{\pi} \text{kei}_2 \bar{x} \right) + \left(1 - \frac{a}{b}\right) \left(\text{bei}_2 \bar{x} - \frac{2}{\pi} \text{ker}_2 \bar{x} \right) \right] \\ & \left. + A_4 \left[- \left(1 - \frac{a}{b}\right) \left(\text{ber}_2 \bar{x} + \frac{2}{\pi} \text{kei}_2 \bar{x} \right) - \left(1 + \frac{a}{b}\right) \left(\text{bei}_2 \bar{x} - \frac{2}{\pi} \text{ker}_2 \bar{x} \right) \right] \right\}\end{aligned}$$

Remembering that

$$\bar{V} = \sqrt{h^3 y} V, \quad \bar{U} = \sqrt{\frac{y}{h}} U$$

$$\eta = \sqrt{h_0 y_0} \left(1 + \frac{\bar{x}^2 - 4K}{4\beta} \right)$$

$$\xi = \frac{2}{\bar{x}} + \frac{\bar{x}(1-2\nu)}{4\beta \left(1 + \frac{\bar{x}^2 - 4K}{4\beta} \right)}$$

$$\psi = \frac{2}{\bar{x}} - \frac{\bar{x}(1+2\nu)}{4\beta \left(1 + \frac{\bar{x}^2 - 4K}{4\beta} \right)}$$

and, re-arranging the expressions with four new constants to be consistant in form with case I, the quantities of interest become

$$V = \frac{4K}{\sqrt{2} b h_0 \bar{x}^2 \eta} [C_1 \text{bei}_2 \bar{x} - C_2 \text{ber}_2 \bar{x} + C_3 \text{kei}_2 \bar{x} - C_4 \text{ker}_2 \bar{x}]$$

$$\begin{aligned}
U = & \frac{-h_0 \bar{x}^2}{4\sqrt{2} K \eta} \left[C_1 \left(\text{ber}_2 \bar{x} + \frac{a}{b} \text{bei}_2 \bar{x} \right) \right. \\
& + C_2 \left(\text{bei}_2 \bar{x} - \frac{a}{b} \text{ber}_2 \bar{x} \right) \\
& + C_3 \left(\text{ker}_2 \bar{x} + \frac{a}{b} \text{kei}_2 \bar{x} \right) \\
& \left. + C_4 \left(\text{kei}_2 \bar{x} - \frac{a}{b} \text{ker}_2 \bar{x} \right) \right]
\end{aligned}$$

$$\begin{aligned}
M_y = & \frac{E h_0 \rho \bar{x}^3}{96 \sqrt{2} (1 - \nu^2) K b \eta} \left[C_1 (\xi \text{bei}_2 \bar{x} - \text{bei}'_2 \bar{x}) \right. \\
& + C_2 (-\xi \text{ber}_2 \bar{x} + \text{ber}'_2 \bar{x}) \\
& + C_3 (\xi \text{kei}_2 \bar{x} - \text{kei}'_2 \bar{x}) \\
& \left. + C_4 (-\xi \text{ker}_2 \bar{x} + \text{ker}'_2 \bar{x}) \right]
\end{aligned}$$

$$\begin{aligned}
\delta = & \frac{\sqrt{2} \rho K \eta \sin \alpha}{E h_0^2 \bar{x}} \left\{ C_1 \left[\psi \left(\text{ber}_2 \bar{x} + \frac{a}{b} \text{bei}_2 \bar{x} \right) + \text{ber}'_2 \bar{x} + \frac{a}{b} \text{bei}'_2 \bar{x} \right] \right. \\
& + C_2 \left[\psi \left(\text{bei}_2 \bar{x} - \frac{a}{b} \text{ber}_2 \bar{x} \right) + \text{bei}'_2 \bar{x} - \frac{a}{b} \text{ber}'_2 \bar{x} \right] \\
& + C_3 \left[\psi \left(\text{ker}_2 \bar{x} + \frac{a}{b} \text{kei}_2 \bar{x} \right) + \text{ker}'_2 \bar{x} + \frac{a}{b} \text{kei}'_2 \bar{x} \right] \\
& \left. + C_4 \left[\psi \left(\text{kei}_2 \bar{x} - \frac{a}{b} \text{ker}_2 \bar{x} \right) + \text{kei}'_2 \bar{x} - \frac{a}{b} \text{ker}'_2 \bar{x} \right] \right\}
\end{aligned}$$

II. AXISYMMETRIC PRESSURE AND VERTICAL EDGE FORCES

The following is an approximate solution for a conical shell of the type described in Part I. This solution is for uniform pressure loads or vertical edge forces, as shown in Fig. 2. The differential equations differ from those of Part I in that the loading term makes the equations nonhomogeneous. The solution, therefore, can be divided into a homogeneous solution (identical to that of Part I), and a supplementary solution for the loading term.

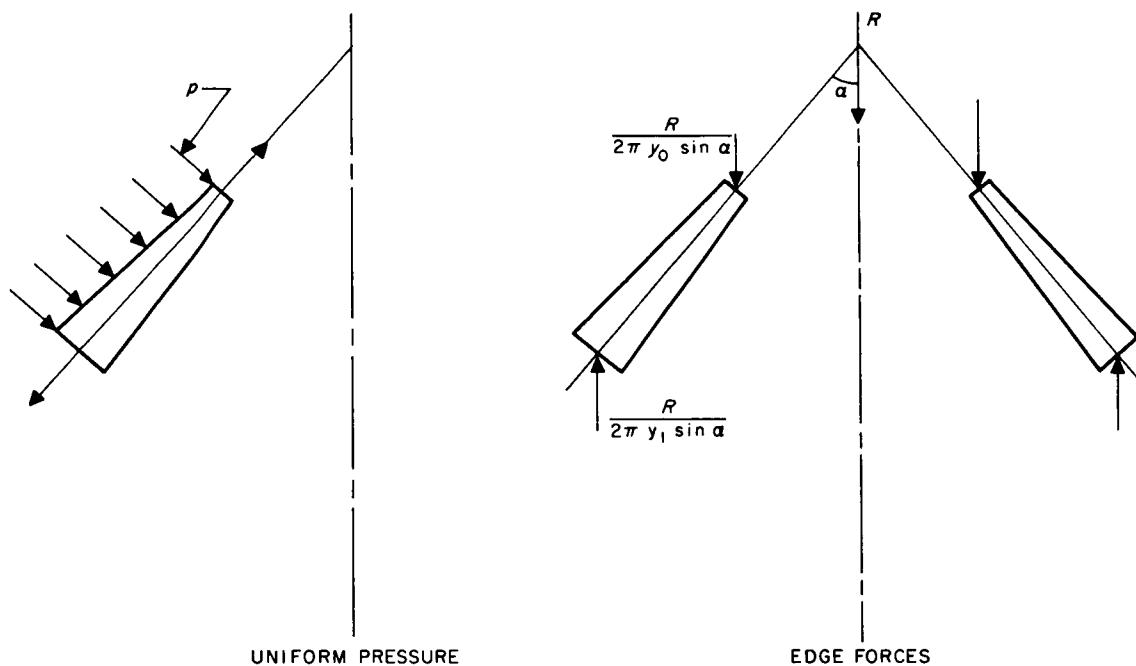


Fig. 2. Applied forces

A. Uniform Pressure Loading

The equations used in the derivation of Eqs. (1) and (2) that are affected by the surface load p are the following (Ref. 1, Sections 127, 128):

$$N_\phi r_0 + N_\theta r_1 \sin \phi + \frac{d}{d\phi} (Q r_0) + r_0 r_1 p = 0$$

$$2\pi r_0 N_\phi \sin \phi + 2\pi r_0 Q_\phi \cos \phi = -p \pi r_0^2$$

Equations (5) and (6) become

$$\frac{d^2\bar{U}}{dy^2} + \bar{U} \left[-\frac{3}{4} \left(\frac{h'}{h} \right)^2 - \frac{3}{4y^2} + \frac{h'}{hy} \left(\frac{1}{2} + \nu \right) \right] + p \sqrt{\frac{y}{h}} \tan \alpha \left[\frac{3}{2} - \gamma \frac{h'}{h} \left(1 - \frac{\nu}{2} \right) \right] = \frac{E\bar{V}}{hy \tan \alpha}$$

$$\frac{d^2\bar{V}}{dy^2} + \bar{V} \left[-\frac{3}{4} \left(\frac{h'}{h} \right)^2 - \frac{3}{4y^2} - \frac{3h'}{hy} \left(\frac{1}{2} - \nu \right) \right] = -\frac{\bar{U} 12(1-\nu^2)}{Ehy \tan \alpha}$$

Following the same procedure as in Part I, Eq. (10) becomes

$$x^2 \frac{d^2W}{dx^2} + \left[-\frac{3}{4} + xiK \right] W = \epsilon_p x^{3/2} + \gamma_p x^{1/2} \quad (16)$$

where

$$\epsilon_p = -\frac{3}{2} p \frac{h_0}{\rho^2} \sqrt{h_0 y_0} \tan \alpha$$

$$\gamma_p = p \frac{y_0}{\rho} \sqrt{h_0 y_0} \left(1 - \frac{\nu}{2} \right) \tan \alpha$$

The solution for the homogeneous part of Eq. (16) is the same as the solution given in Part I, namely Eq. (12) or (15), depending on the sign of K . The particular solution is of the form

$$W_p = -\frac{\epsilon_p}{K^2} x^{-1/2} - i \left(\frac{\epsilon_p}{K} x^{1/2} + \frac{\gamma_p}{K} x^{-1/2} \right)$$

Hence,

$$\bar{V}_p = -\frac{1}{bK} \left(\frac{\epsilon_p x + \gamma_p}{x^{1/2}} \right)$$

$$\bar{U}_p = \frac{1}{Kx^{1/2}} \left[-\frac{\epsilon_p}{K} + \frac{a}{b} (\epsilon_p x + \gamma_p) \right]$$

The complete solution consists of the particular solution plus the homogeneous or bending solution. Letting the subscript b denote the bending solution, the quantities of interest become:

Case I, $h_1 < h_0$

$$V = V_b + \frac{16}{b\bar{x}^4 h_0 \eta} \left(\epsilon_p \frac{\bar{x}^2}{4} + \gamma_p k \right)$$

$$U = U_b - \frac{h_0}{k^2 \eta} \left[\epsilon_p + \frac{a}{b} \left(\epsilon_p \frac{\bar{x}^2}{4} + \gamma_p k \right) \right]$$

$$M_y = M_{y_b} + \frac{Eh_0 \rho \bar{x}}{24(1-\nu^2)k^2 b \eta} \left[\left(\xi + \frac{2}{\bar{x}} \right) \left(\epsilon_p \frac{\bar{x}^2}{4} + \gamma_p k \right) - \frac{\epsilon_p \bar{x}}{2} \right]$$

$$\delta = \delta_b + \frac{4 \eta^2 k \sin \alpha}{Eh_0^2 \bar{x}^2} \left\{ \left[- \frac{h_0^2 (1+2\nu)}{2k^2 \eta^3} \right] \left[\epsilon_p + \frac{a}{b} \left(\epsilon_p \frac{\bar{x}^2}{4} + \gamma_p k \right) \right] + \frac{a \rho \epsilon_p}{b \eta k} - \frac{p \eta^2}{h_0} \left(1 - \frac{\nu}{2} \right) \tan \alpha \right\}$$

Case II, $h_1 > h_0$

$$V = V_b - \frac{16}{b\bar{x}^4 h_0 \eta} \left(\epsilon_p \frac{\bar{x}^2}{4} + \gamma_p K \right)$$

$$U = U_b + \frac{h_0}{K^2 \eta} \left[- \epsilon_p + \frac{a}{b} \left(\epsilon_p \frac{\bar{x}^2}{4} + \gamma_p K \right) \right]$$

$$M_y = M_{y_b} - \frac{Eh_0 \rho \bar{x}}{24(1-\nu^2)K^2 b \eta} \left[\left(\xi + \frac{2}{\bar{x}} \right) \left(\epsilon_p \frac{\bar{x}^2}{4} + \gamma_p K \right) - \epsilon_p \frac{\bar{x}}{2} \right]$$

$$\delta = \delta_b + \frac{4 \eta^2 K \sin \alpha}{Eh_0^2 \bar{x}^2} \left\{ \left[\frac{h_0^2 (1+2\nu)}{2K^2 \eta^3} \right] \left[- \epsilon_p + \frac{a}{b} \left(\epsilon_p \frac{\bar{x}^2}{4} + \gamma_p K \right) \right] - \frac{a \rho \epsilon_p}{b K \eta} - \frac{p \eta^2}{h_0} \left(1 - \frac{\nu}{2} \right) \tan \alpha \right\}$$

where

$$\delta = \frac{\gamma \sin \alpha}{Eh} \left[-\frac{dU}{dy} + \nu \frac{U}{y} - py \left(1 - \frac{\nu}{2} \right) \tan \alpha \right]$$

The bending solution contains four constants. These can be evaluated by using the following boundary conditions:

At

$$y = y_0$$

$$M = 0$$

$$Q = 0$$

At

$$y = y_1$$

$$M = 0$$

$$Q = 0$$

where

$$M = M_b + M_p$$

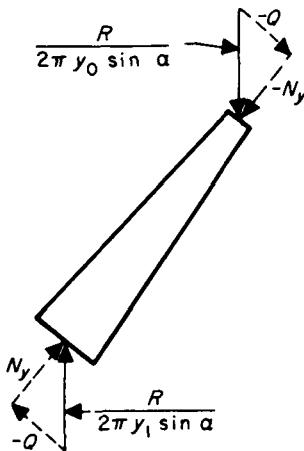
$$Q = Q_b + Q_p$$

B. Vertical Load R

Let R be the total resultant vertical force acting on the shell. Assume that R is uniformly distributed around the edge of the shell, as shown in Fig. 3. The same equations are affected by the vertical load as by the internal pressure.

$$N_\phi r_0 + N_\theta r_1 \sin \phi + \frac{d}{d\phi} (Q r_0) = 0$$

$$2\pi r_0 N_\phi \sin \phi + 2\pi r_0 Q_\phi \cos \phi = -R$$

Fig. 3. Components of vertical force R

Upon substitution and change of variables, Eq. (10) becomes

$$x^2 \frac{d^2 W}{dx^2} + \left[-\frac{3}{4} + xiK \right] = \epsilon_R x^{3/2} + \gamma_R x^{1/2} \quad (17)$$

where

$$\epsilon_R = \frac{R h_0^2}{2 \pi y_0 \rho^2 \sin \alpha \cos \alpha \sqrt{y_0 h_0}}, \quad \gamma_R = \frac{-\nu R h_0}{2 \rho \pi \sqrt{y_0 h_0} \sin \alpha \cos \alpha}$$

Equation (17) is identical to Eq. (16), except for the values of ϵ and γ , which are constant. Therefore, the solution for every quantity except δ is the same as that for the uniform pressure loading, except that one must replace ϵ_p and γ_p with ϵ_R and γ_R , respectively.

The quantities of interest become

Case I, $h_1 < h_0$

$$V = V_b + \frac{16}{b \bar{x}^4 h_0 \eta} \left(\epsilon_R \frac{\bar{x}^2}{4} + \gamma_R k \right)$$

$$U = U_b - \frac{h_0}{k^2 \eta} \left[\epsilon_R + \frac{a}{b} \left(\epsilon_R \frac{\bar{x}^2}{4} + \gamma_R k \right) \right]$$

$$M_y = M_{y_b} + \frac{Eh_0 \rho \bar{x}}{24(1-\nu^2)k^2 b \eta} \left[\left(\xi + \frac{2}{\bar{x}} \right) \left(\epsilon_R \frac{\bar{x}^2}{4} + \gamma_R k \right) - \epsilon_R \frac{\bar{x}}{2} \right]$$

$$\delta = \delta_b + \frac{4\eta^2 k \sin \alpha}{Eh_0^2 \bar{x}^2} \left\{ \left[-\frac{h_0^2 (1+2\nu)}{2k^2 \eta^3} \right] \left[\epsilon_R + \frac{a}{b} \left(\epsilon_R \frac{\bar{x}^2}{4} + \gamma_R k \right) \right] + \frac{a \rho \epsilon_R}{b \eta k} + \frac{h_0 \nu R}{2\pi \eta^2 \cos \alpha \sin \alpha} \right\}$$

Case II, $h_1 > h_0$

$$V = V_b - \frac{16}{b \bar{x}^4 h_0 \eta} \left(\epsilon_R \frac{\bar{x}^2}{4} + \gamma_R K \right)$$

$$U = U_b + \frac{h_0}{K^2 \eta} \left[-\epsilon_R + \frac{a}{b} \left(\epsilon_R \frac{\bar{x}^2}{4} + \gamma_R K \right) \right]$$

$$M_y = M_{y_b} - \frac{Eh_0 \rho \bar{x}}{24(1-\nu^2)K^2 b \eta} \left[\left(\xi + \frac{2}{\bar{x}} \right) \left(\epsilon_R \frac{\bar{x}^2}{4} + \gamma_R K \right) - \epsilon_R \frac{\bar{x}}{2} \right]$$

$$\delta = \delta_b + \frac{4\eta^2 K \sin \alpha}{Eh_0^2 \bar{x}^2} \left\{ \left[-\frac{h_0^2 (1+2\nu)}{2K^2 \eta^3} \right] \left[-\epsilon_R + \frac{a}{b} \left(\epsilon_R \frac{\bar{x}^2}{4} + \gamma_R K \right) \right] - \frac{a \rho \epsilon_R}{b K \eta} + \frac{h_0 \nu R}{2\pi \eta^2 \cos \alpha \sin \alpha} \right\}$$

where

$$\delta = \frac{y \sin \alpha}{Eh} \left[-\frac{dU}{dy} + \frac{\nu U}{y} + \frac{\nu R}{2\pi y \sin \alpha \cos \alpha} \right]$$

The boundary conditions needed to evaluate the four constants in the bending solution are different from those of the uniform pressure case. Since R is a vertical force acting on the edge of the shell, we can say the following:

At

$$\gamma = \gamma_0$$

$$M = 0$$

$$H = 0$$

At

$$\gamma = \gamma_1$$

$$M = 0$$

$$H = 0$$

where

$$M = M_b + M_p$$

H = the total horizontal force per unit length

$$-N_y \cos \alpha - Q_y \sin \alpha = \frac{R}{2\pi\gamma_0 \sin \alpha}$$

$$-N_y \sin \alpha + Q_y \cos \alpha = 0$$

or,

$$-N_y \cos \alpha \sin \alpha - Q_y \sin^2 \alpha = \frac{R}{2\pi\gamma_0}$$

$$-N_y \sin \alpha \cos \alpha + Q_y \cos^2 \alpha = 0$$

$$Q_{\gamma_0} = \frac{-R}{2\pi\gamma_0}$$

Similarly,

$$Q_{\gamma_1} = \frac{-R}{2\pi\gamma_1}$$

Thus, the boundary conditions become:

at

$$\gamma = \gamma_0$$

$$M = 0$$

$$Q = \frac{-R}{2\pi\gamma_0}$$

at

$$\gamma = \gamma_1$$

$$M = 0$$

$$Q = \frac{-R}{2\pi\gamma_1}$$

where

$$M = M_b + M_R$$

$$Q = Q_b + Q_R$$

III. DISCUSSION

The one condition made to simplify the differential equations is that Δ/y_0 be small compared to one. The approximation is independent of the opening angle of the cone and also of the behavior of the derivatives of W involved. Therefore, the solution should be applicable for shallow or steep cones as long as Δ/y_0 is small. As the cone approaches a cylinder ($\alpha = 0^\circ$), the solution reduces to that of a tapered cylindrical shell (Ref. 4). For shallow cones, the solution applies if

$$\frac{12(1+\nu)}{\rho^2(1-\nu)\tan^2\alpha} > 1$$

If

$$\frac{12(1+\nu)}{\rho^2(1-\nu)\tan^2\alpha} \leq 1$$

this analysis is not applicable, since the argument \bar{x} and other parameters become zero or complex.

NOMENCLATURE

$$a = \frac{E \rho \tan \alpha}{12(1 + \nu)}$$

$$b = \frac{E \rho \tan \alpha}{12(1 + \nu)} \sqrt{\frac{12(1 + \nu)}{(1 - \nu) \rho^2 \tan^2 \alpha} - 1}$$

C_1, C_2, C_3, C_4 constants of integration

$$D = \frac{E h^3}{12(1 - \nu^2)}$$

E Young's modulus

h shell thickness

h' derivative of h with respect to distance

H horizontal force, lb/in.

i $\sqrt{-1}$

$$K = \frac{h_0(1 - \nu)}{\rho \gamma_0} \sqrt{\frac{12(1 + \nu)}{(1 - \nu) \rho^2 \tan^2 \alpha} - 1}$$

k $|K|$

M_y meridional bending moment, lb/in./in.

N_y meridional membrane force, lb/in.

n nondimensional thickness gradient

p pressure, psi

Q_y transverse shear, lb/in.

r radius of curvature

R resultant vertical force, lb

\bar{s} nondimensional coordinate

NOMENCLATURE (Cont'd)

$$U \quad Q_y r_2$$

$$\bar{U} \quad U \sqrt{\frac{y}{h}}$$

V rotation of meridian

$$\bar{V} \quad \sqrt{h^3 y} V$$

$$\bar{W} \quad U + \lambda \bar{V}$$

x nondimensional coordinate

$$\bar{x} \quad 2\sqrt{k} x^{1/2}$$

y coordinate

α cone opening angle

$$\beta \quad \frac{\rho \gamma_0}{h_0} k$$

Δ meridional length of shell, $\gamma_1 - \gamma_0$

$$\epsilon_p \quad - \frac{3}{2} p \frac{h_0}{\rho^2} \sqrt{h_0 \gamma_0} \tan \alpha$$

$$\epsilon_R \quad \frac{R h_0^2}{2\pi y_0 \rho^2 \sin \alpha \cos \alpha \sqrt{y_0 h_0}}$$

$$\gamma_p \quad \frac{p \gamma_0}{\rho} \sqrt{h_0 \gamma_0} \left(1 - \frac{\nu}{2} \right) \tan \alpha$$

$$\gamma_R \quad - \frac{\nu h_0 R}{2\pi \rho \sqrt{h_0 \gamma_0} \sin \alpha \cos \alpha}$$

λ complex constant, $a + ib$

NOMENCLATURE (Cont'd)

$$\eta = \sqrt{h_0 \gamma_0 \left(1 + \frac{\bar{x}^2 - 4k}{4\beta} \right)}$$

$$\rho = \text{thickness gradient, } \frac{h_1 - h_0}{\gamma_1 - \gamma_0}$$

ϕ polar angle

ν Poisson's ratio

$$\psi = \frac{2}{\bar{x}} - \frac{\bar{x} (1 + 2\nu)}{4\beta \left(1 + \frac{\bar{x}^2 - 4k}{4\beta} \right)}$$

$$\xi = \frac{2}{\bar{x}} + \frac{\bar{x} (1 - 2\nu)}{4\beta \left(1 + \frac{\bar{x}^2 - 4k}{4\beta} \right)}$$

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